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On the Non-Commuting Graph of the Group U_{6n}

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Abstract

Let G be a finite group. The non-commuting graph of G is a simple graph $\Gamma(G)$ whose vertices are elements of $G\backslash Z(G)$, where $Z(G)$ is the center of G, and two distinct vertices a and b are joint by an edge if $ab \neq ba$. In this paper, we study the non-commuting graph of the group U_{6n}. The independent number, clique and chromatic numbers of the non-commuting graph of the group U_{6n}, $\Gamma(U_{6n})$, are determined. Additionally, the resolving polynomial, total eccentricity and independent polynomials of $\Gamma(U_{6n})$ are computed. Finally, the detour and eccentric connectivity indices of $\Gamma(U_{6n})$ are found.

Keywords: non-commuting graph; independent number; chromatic number; clique number; resolving polynomial of a graph.

1 Introduction

In the last three decades, the interrelation of the structure in graphs and algebra has provided us some interesting results and the topic has earned significant attention from the research community, for example, see [5, 17].

For a finite group G, the center of G and the centralizer of $a \in G$ are denoted by $Z(G)$ and $C_G(a)$, respectively. The *AC-group* G is a group such that $C_G(a)$ is abelian for each $a \in G\backslash Z(G)$ [17]. The Hungarian mathematician Paul Erdös [18] introduced the notion of the *non-commuting graph* of a group in the way that the vertices are the non-central elements of a group and two distinct vertices are joint by an edge if they do not commute in the group. He posed the problem that for a group G whose non-commuting graph has no infinite complete subgraph, is it true that there is a finite bound on the cardinalities of complete subgraphs of $\Gamma(G)$? Neumann [18] answered Erdös's question positively. Since then, the topic has been widely studied by researchers in the field, Abdollahi et al. [1] explored how the graph theoretical properties of $\Gamma(G)$ can affect the group theoretical properties of G, while Vatandoost and Khalili [25] found the bounds of domination number of the non-commuting graph of a finite groups. Furthermore, Romdhini et al. [20] studied the neighbor degree sum energy of the non-commuting graph for dihedral groups. It is worth noting that if the underlying group is abelian, then the non-commuting graph has no element since in that case the group is equal to its center. In this paper, we consider the non-abelian group U_{6n}, which will be defined later. As a matter of fact there are some researches studying commuting graph of groups, for instance [3, 4], and there are some researches on the commuting graph of the group U_{6n}, see [10, 22, 11]. In addition, there are several articles about the non-commuting graph of a group, for instance, Abdollahi et al. [1] investigated on the non-commuting graph of finite groups whereas Talebi [24] has conducted the same investigation for the dihedral groups. Furthermore, the energy of the non-commuting graph of the group U_{6n} has been studied in [12, 23].

Most recently, Khasraw et al. [16], considered the non-commuting graph, $\Gamma(D_{2n})$ of dihedral group. They found the detour index, eccentric connectivity and the total eccentricity polynomials, and the mean distance of $\Gamma(D_{2n})$.

Some fundamental concepts that are related to this research are provided in what follows. Throughout the paper, all graphs are assumed to be simple, that is, there is no loops and multiple edges. By a *finite graph* we mean a graph in which vertex set and edge set are finite. Hence, the vertex-set and the edge-set of the graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively, while we denote the size of $V(\Gamma)$ by $n(\Gamma)$ and the size of $E(\Gamma)$ by $|E(\Gamma)|$.

Let $k \geq 2$. A sequence of k vertices in which every vertex in the sequence is linked to a vertex next to it is known as a *path* of a graph, denoted by P_k. A *simple path* is a path that has no repeated vertices. A path that starts and ends at the same vertex is referred to as a *circuit* while any circuit that does not repeat vertices is a *cycle*, denoted as C_n, where $n \geq 3$ [5, 7].

A graph is called *connected* if for any two vertices u and v, the path from u to v exists, while a *disconnected* graph consists of connected pieces called *components*. If vertices u and v are connected in Γ, the *distance* (*detour distance*) between u and v, will be denoted by $d(u, v)(D(u, v))$, is the length of a shortest (longest) path from u to v in Γ. For a given vertex v in Γ, the maximum distance between v and any other vertex in Γ is called the *eccentricity* of v, denoted by $ecc(v)$. The *degree* of a vertex v, $deg(v)$, is the number of edges incident with v [5].

For a graph Γ, the polynomials $D(\Gamma, q) = \sum_{u,v \in V(\Gamma)} q^{D(u,v)}$ [21], $\Xi(\Gamma, q) = \sum_{u \in V(\Gamma)} deg_\Gamma(u)$

$q^{ecc(u)}$ and $\Theta(\Gamma, q) = \sum_{u \in V(\Gamma)} q^{ecc(u)}$ [9] are called the *detour*, *eccentric connectivity* and *total eccentricity polynomials* , respectively. The first derivative of $D(\Gamma, q)$ at 1 is called the *detour index* of the graph Γ, and denoted by $dd(\Gamma)$.

By a *regular graph* we mean a graph in which all of its vertices have the same degree and a graph is called n*-regular* if all of its vertices have degree n [5]. While the *chromatic number* of a graph Γ, $\chi(\Gamma)$, is defined as the minimum number c for which is c*-vertex* colorable [13]. The *clique number* of a graph Γ, $\omega(\Gamma)$, is defined to be the size of the largest complete subgraph of Γ. A *vertex cover* of a graph Γ is a subset S of $V(\Gamma)$ for which every edge of Γ has at least one vertex in S. The minimum size of a vertex cover is denoted by $\tau(\Gamma)$ [19]. A non-empty set S of $V(\Gamma)$ is called *independent* if no two vertices in S are adjacent in Γ. The *independent number* is the cardinality of a maximum independent set of a graph Γ and is denoted by $\alpha(\Gamma)$ [5]. Let Γ be a graph. The *independent polynomial* was defined in [14] as follows: $I(\Gamma, q) = \sum_{i=0}^{\alpha(\Gamma)} s_i q^i$; where s_i is the number of independent sets of Γ of cardinality i. While, in [8], the *vertex-cover polynomial* is defined as $\psi(\Gamma, q) = \sum_{i=0}^{n(\Gamma)} c_i q^i$; where c_i is the number of vertex covers of Γ of cardinality i.

For an integer $k \geq 2$, a graph Γ is called k*-partite* if $V(\Gamma)$ can be partitioned into k classes such that the end vertices of each edge lie in different classes, and no two vertices in the same class are adjacent. If any two vertices from different classes are adjacent, then the graph is called *complete* k*-bipartite graph*, and denoted by $K_{r_1, r_2, \cdots, r_k}$.

Let Γ be a graph. Suppose $W = \{w_1, w_2, \cdots, w_k\}$, where $k \leq n(\Gamma)$, is a subset of $V(\Gamma)$. The *representation* of a vertex v of Γ is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$. If every pair of distinct vertices of Γ have distinct representations with respect to W, then W is called a *resolving set* for Γ. The cardinality of a minimum resolving set for Γ is called the *metric dimension* of Γ, denoted by $\beta(\Gamma)$ [6]. The *resolving polynomial* of a graph Γ, denoted by $\beta(\Gamma, q)$, is defined by $\beta(\Gamma, q) = \sum_{i=\beta(\Gamma)}^{n(\Gamma)} r_i q^i$, where r_i is the number of resolving sets for Γ of cardinality i. The sequence $(r_{\beta(\Gamma)}, r_{\beta(\Gamma)+1}, \cdots, r_{n\Gamma})$ is called the *resolving sequence*. The set of all distinct roots of $\beta(\Gamma, q)$ is denoted by $Z(\beta(\Gamma, q))$.

The group U_{6n}, of order $6n$, is defined by

$$U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle,$$

for $n \geq 1$ with center $Z(U_{6n}) = \langle a^2 \rangle$ [15]. Throughout this paper, the elements of $U_{6n} \backslash Z(U_{6n})$ are partitioned into four disjoint sets according to centralizers of elements, see Lemma 2.1, as follows: $\Omega_1 = \{a^{2r+1} : 0 \leq r \leq n-1\}$, $\Omega_2 = \{a^{2r+1}b : 0 \leq r \leq n-1\}$, $\Omega_3 = \{a^{2r+1}b^2 : 0 \leq r \leq n-1\}$, and $\Omega_4 = \{a^{2r}b^k : 0 \leq r \leq n-1,\ k = 1, 2\}$. It is clear that $|\Omega_1| = |\Omega_2| = |\Omega_3| = n$, and $|\Omega_4| = 2n$. Note that, throughout the paper, $\Gamma(U_{6n})$ indicates the non-commuting graph of the group U_{6n}.

The paper consists of four sections. The first section is introduction, where the necessary concepts are presented. Some basic properties of $\Gamma(U_{6n})$ of U_{6n} are studied in Section 2. In Section 3, we find the resolving polynomial of $\Gamma(U_{6n})$, while in Section 4, the detour index, eccentric connectivity, total eccentricity and independent polynomials of $\Gamma(U_{6n})$ are computed.

2 Some Properties of $\Gamma(U_{6n})$

This section contains some lemmas on non-commuting graph that are used to obtain some important results that follow.

In [17], the following lemma had been shown.

Lemma 2.1. *For the group U_{6n}, and $0 \leq r \leq n-1$, we have the following*

1. $Z(U_{6n}) = \langle a^2 \rangle$,
2. $C_{U_{6n}}(a^{2r+1}) = \langle a \rangle$,
3. $C_{U_{6n}}(a^{2r+1}b) = \langle a^2 \rangle \cdot \langle \{a^{2s+1}b : 0 \leq s \leq n-1\} \rangle$,
4. $C_{U_{6n}}(a^{2r+1}b^2) = \langle a^2 \rangle \cdot \langle \{a^{2s+1}b^2 : 0 \leq s \leq n-1\} \rangle$,
5. $C_{U_{6n}}(a^{2r}b) = \langle a^2 \rangle \cdot \langle \{a^{2s}b, a^{2s}b^2 : 0 \leq s \leq n-1\} \rangle$.

The following useful lemma is used in calculating the degree of vertices in $\Gamma(U_{6n})$, which can be found in [1].

Lemma 2.2. *Suppose G is a non-abelian finite group and let $x \in V(\Gamma(G))$. Then, $\deg(x) = |G| - |C_G(x)|$, where $C_G(x)$ is the centralizer of the element x in G.*

The above lemmas lead to the following.

Corollary 2.1. *Let $n \geq 1$ be an integer and let $\Gamma = \Gamma(U_{6n})$. Then, for $0 \leq r \leq n-1$ and $k = 1, 2$, we have*

1. $\deg_\Gamma(a^{2r+1}) = 4n$,
2. $\deg_\Gamma(a^{2r+1}b^k) = 4n$,
3. $\deg_\Gamma(a^{2r}b^k) = 3n$.

Theorem 2.1. *For $n \geq 1$, $\Gamma = \Gamma(U_{6n})$, $|E(\Gamma)| = 9n^2$.*

Proof. From Corollary 2.1, we have $|E(\Gamma)| = \frac{1}{2}\sum_{x \in V(\Gamma)} \deg(x) = \frac{1}{2}\left(12n^2 + 6n^2\right) = 9n^2$. □

Theorem 2.2. *For $n \geq 1$, let $\Gamma = \Gamma(U_{6n})$ and Ω is a subset of U_{6n}. Then, $\Gamma = K_{n,n,n,2n}$, the 4-partite graph, if and only if $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$.*

Proof. Assume $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. Then, $C_\Omega(a^{2r+1}) = \Omega_1, C_\Omega(a^{2r+1}b) = \Omega_2, C_\Omega(a^{2r+1}b^2) = \Omega_3$ and $C_\Omega(a^{2r}b^k) = \Omega_4$ for $k = 1, 2$, so $\Gamma = K_{n,n,n,2n}$. Conversely, assume $\Gamma = K_{n,n,n,2n}$, then by Corollary 2.1, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. □

In [5], the relation between the independent number and the vertex cover with the number of vertices has been given as follows.

Lemma 2.3. *Let Γ be a graph. Then, $\alpha(\Gamma) + \tau(\Gamma) = n(\Gamma)$.*

Theorem 2.3. *For the graph $\Gamma = \Gamma(U_{6n})$, $\alpha(\Gamma) = 2n$.*

Proof. From Lemma 2.1 and Theorem 2.2, one can see that Ω_4 is the largest part of the 4-partite graph $K_{n,n,n,2n}$. Thus, $\alpha(\Gamma) = 2n$. □

Corollary 2.2. *For the graph* $\Gamma = \Gamma(U_{6n})$, $\tau(\Gamma) = 3n$.

Proof. The proof is a straightforward from Lemma 2.3 and Theorem 2.3. □

Theorem 2.4. *Let* $\Gamma = \Gamma(U_{6n})$. *Then,* $\chi(\Gamma) = \omega(\Gamma) = 4$.

Proof. By Theorem 2.2, the clique of Γ can only contain one vertex from each $\Omega_i (i = 1, ..., 4)$. Therefore, $\omega(\Gamma) = 4$. Since the group U_{6n} is an AC-group, then $\chi(\Gamma) = \omega(\Gamma)$. □

Theorem 2.5. *Let* $\Gamma = \Gamma(U_{6n})$, *and* $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. *There exist no subset* S *of* Ω *such that* $\Gamma = C_5$.

Proof. Suppose that $\Gamma = C_5$. Then, at least two vertices, say v_1 and v_2, on Γ belong to some Ω_i for $i \in \{1, 2, 3, 4\}$. Three cases have to be considered. **Case 1.** If the other three vertices belong to Ω_j, $j \in \{1, 2, 3, 4\}$ and $j \neq i$, then $\Gamma = K_{2,3}$, which is a contradiction. **Case 2.** If two of the other three vertices belong to the same Ω_j and the third one belongs to Ω_k, where $j \neq i \neq k$, then $\Gamma = K_{2,2,1}$, which is also a contradiction. **Case 3.** If each of the other three vertices belongs to a different Ω_j, $j \in \{1, 2, 3, 4\}$ and $j \neq i$, then $\Gamma = K_{2,1,1,1}$, which is again a contradiction. □

Theorem 2.6. *Let* $\Gamma = \Gamma(U_{6n})$, *and* Ω *is a subset of* $V(\Gamma)$. *Then,* $\Gamma \neq P_k$ *for* $k \geq 4$.

Proof. For $k < 4$, we show that, $\Gamma = P_2$ or P_3.
Case 1. $\Omega = \{x, y\}$ where $x \notin C_{U_{6n}}(y)$, then $\Gamma = P_2$. **Case 2.** Let $\Omega = \{x, y, z\}$ where $x \notin C_{U_{6n}}(y)$. If $z \notin C_{U_{6n}}(x)$ and $z \notin C_{U_{6n}}(y)$ then $\Gamma = C_3$. But if z is either in $C_{U_{6n}}(x)$ or $C_{U_{6n}}(y)$ then $\Gamma = P_3$. If we add one more element, say w, to Ω then we have two possibilities; either $w \notin C_{U_{6n}}(x)$ and $w \notin C_{U_{6n}}(y)$, and this implies that there will be edges $w \sim x$ and $w \sim y$, which means $\Gamma \neq P_4$, or w is in the centralizer of one of them, in this case say $w \in C_{U_{6n}}(x)$, implies that there will be an edge $w \sim y$, which again means $\Gamma \neq P_4$. □

Theorem 2.7. *Let* $\Gamma = \Gamma(U_{6n})$, *and let* Ω *be a subset of* $V(\Gamma)$. *Then,* Γ *is* $2n$*-regular if and only if* $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$.

Proof. If the graph Γ is $2n$-regular, then every vertex in Γ has degree $2n$. By Corollary 2.1, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Conversely, let $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. By Corollary 2.1, $deg(u) = 2n$ for all $u \in \Omega$. Thus, Γ is a $2n$-regular graph. □

3 Resolving Polynomial of $\Gamma(U_{6n})$

The main aim of this section is to determine the metric dimension and resolving polynomial of $\Gamma(U_{6n})$. We start by the following useful lemma about resolving polynomial $\beta(\Gamma, q)$ of a graph Γ of order n.

Lemma 3.1. *Let Γ be a connected graph such that $n(\Gamma) = n$. Then, Γ has only one resolving set of cardinality n, which is $V(\Gamma)$, and n resolving sets of cardinality $n-1$.*

Theorem 3.1. *Let $\Gamma = \Gamma(U_{6n})$. Then,*

$$\beta(\Gamma) = \begin{cases} 3 & \text{if } n = 1, \\ 5n-4 & \text{if } n > 1. \end{cases}$$

Proof. **Case 1**. When $n = 1$. The graph Γ is split with 5 vertices such that $V(\Gamma) = K \cup S$, where $K = \{a, ab, ab^2\}$ is the complete part and $S = \{b, b^2\}$ is the independent part. The resolving set for Γ of minimal cardinality is $W = \{a, ab, b\}$. **Case 2**. When $n > 1$. Since every two distinct vertices u and v are non-adjacent in Ω_i, $i \in \{1, 2, 3, 4\}$, then $\beta(\Gamma) \geq 5n-4$. On the other hand, it is clear that the set $W = \{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2;\ 0 \leq r \leq n-2\} \cup \Omega_4 \backslash \{x\}$, where x is any particular element in Ω_4, is the resolving set for Γ of cardinality $5n-4$. This implies that $\beta(\Gamma) \leq 5n-4$. ☐

Theorem 3.2. *Let $\Gamma = \Gamma(U_{6n})$. Then,*

$$\beta(\Gamma, q) = \begin{cases} q^3(q+2)(q+3) & \text{if } n = 1, \\ q^{5n-4}(q+n)^3(q+2n) & \text{if } n > 1. \end{cases}$$

Proof. For n, there are two cases to be considered:

- **When** $n = 1$. Then, $\Gamma = \Gamma(U_6)$. By Theorem 3.1, we need to compute the resolving sequence (r_3, r_4, r_5) of length 3. Since the graph Γ is split with 5 vertices where its complete part consists of 3 vertices and the independent part consists of 2 vertices, then $r_3 = \binom{2}{1}\binom{3}{2} = 6$. By Lemma 3.1, $r_4 = 5$ and $r_5 = 1$.
- **When** $n > 1$. By Theorem 2.2, the graph Γ is 4-partite. By Theorem 3.1, we need to determine the resolving sequence $(r_{5n-4}, r_{5n-3}, r_{5n-2}, r_{5n-1}, r_{5n})$ of length 5.

For r_{5n-4}: By Theorem 2.2 and by the multiplication's principal,

$$r_{5n-4} = \binom{n}{n-1}\binom{n}{n-1}\binom{n}{n-1}\binom{2n}{2n-1} = 2n^4.$$

For r_{5n-3}: It is required to compute all the resolving sets for Γ of cardinality $5n-3$. There are four cases, in the first case, $\binom{n}{n}\binom{n}{n-1}\binom{n}{n-1}\binom{2n}{2n-1}$; in the second case, $\binom{n}{n-1}\binom{n}{n}\binom{n}{n-1}\binom{2n}{2n-1}$; in the third case, $\binom{n}{n-1}\binom{n}{n-1}\binom{n}{n}\binom{2n}{2n-1}$; and in the fourth case, $\binom{n}{n-1}\binom{n}{n-1}\binom{n}{n-1}\binom{2n}{2n}$ possible resolving sets of cardinality $5n-3$. By the addition's principal, $r_{5n-3} = 7n^3$.

For r_{5n-2}: We need to compute all the resolving sets for Γ of cardinality $5n-2$. Again, we have six cases to consider, in the first case, $\binom{n}{n}\binom{n}{n}\binom{n}{n-1}\binom{2n}{2n-1}$; in the second case, $\binom{n}{n}\binom{n}{n-1}\binom{n}{n}\binom{2n}{2n-1}$; in the third case, $\binom{n}{n}\binom{n}{n-1}\binom{n}{n-1}\binom{2n}{2n}$; in the fourth case, $\binom{n}{n-1}\binom{n}{n}\binom{n}{n}\binom{2n}{2n-1}$; in the fifth case, $\binom{n}{n-1}\binom{n}{n}\binom{n}{n-1}\binom{2n}{2n}$; and in the sixth case, $\binom{n}{n-1}\binom{n}{n-1}\binom{n}{n}\binom{2n}{2n-1}$; possible resolving sets of cardinality $5n-2$. By the addition's principal, $r_{5n-2} = 9n^2$. By Lemma 3.1, $r_{5n-1} = 5n$ and $r_{5n} = 1$. ☐

Corollary 3.1. *Let* $\Gamma = \Gamma(U_{6n})$*. Then, for* $n = 1$*,* $Z(\beta(\Gamma, q)) = \{0, -2, -3\}$*, and for all* $n > 1$*,* $Z(\beta(\Gamma, q)) = \{0, -n, -2n\}$*.*

4 Some Polynomials of $\Gamma(U_{6n})$

In this section some properties of non-commuting graphs of U_{6n} are explored, namely the detour, the eccentric connectivity, the total eccentricity and the independent polynomials.

Lemma 4.1. *Let* $\Gamma(U_{6n})$*. Then,* $D(u, v) = 5n - 1$ *for any* $u, v \in V(\Gamma)$*.*

Proof. From Theorem 2.2, one can see that no two vertices in Ω_i are adjacent, and every vertex in Ω_i is adjacent to each vertex in Ω_j for $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$. Then, for all $u, v \in \Omega$, where $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, there is a path of length $5n - 1$ from u to v. ☐

Theorem 4.1. *Let* $\Gamma = \Gamma(U_{6n})$*. Then,* $D(\Gamma, q) = \frac{5n(5n-1)}{2} q^{5n-1}$*.*

Proof. We have that $n(\Gamma) = 5n$. Then, there are $\binom{5n}{2} = \frac{5n(5n-1)}{2}$ possibilities of choosing any two distinct vertices from Γ. By Lemma 4.1, $D(u, v) = 5n - 1$ for any distinct pairs of $u, v \in V(\Gamma)$. Thus, $D(\Gamma, q) = \sum_{\{u,v\}} q^{D(u,v)} = \binom{5n}{2} q^{5n-1} = \frac{5n(5n-1)}{2} q^{5n-1}$. ☐

The direct result from Theorem 4.1 is the following.

Corollary 4.1. *Let* $\Gamma(U_{6n})$*. Then,* $dd(\Gamma(U_{6n})) = \frac{5n(5n-1)^2}{2}$*.*

Lemma 4.2. *Let* $\Gamma(U_{6n})$*. Then,* $ecc(u) = 2$ *for every* $u \in \Omega$*, where* $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$*.*

Proof. In Ω_i, there is no edge between any pair of distinct vertices, for $i \in \{1, 2, 3, 4\}$. Furthermore, every vertex in Ω_i is adjacent to each vertex in Ω_j, for $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$. Then, the maximum distance between any vertex in Ω_i and other vertices in Ω is 2. Therefore, $ecc(u) = 2$, for every $u \in \Omega$. ☐

Theorem 4.2. *Let* $\Gamma = \Gamma(U_{6n})$ *and* $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$*. Then,*

1. $\Theta(\Gamma, q) = 5nq^2$.
2. $\Xi(\Gamma, q) = 18n^2q^2$.

Proof. Since the graph Γ has $5n$ vertices, then

1. By Lemma 4.2, $ecc(u) = 2$, for every $u \in \Omega$, so $\Theta(\Gamma, q) = \sum_{u \in V(\Gamma)} q^{ecc(u)} = 5nq^2$.
2. By Corollary 2.1, $3n$ vertices in $\Omega_1 \cup \Omega_2 \cup \Omega_3$ are of degree $4n$ and $2n$ vertices in Ω_4 are of degree $3n$, and from Lemma 4.2, we see that $\Xi(\Gamma, q) = \sum_{u \in V(\Gamma)} deg_\Gamma(u) q^{ecc(u)} = (3n(4n) + 2n(3n))q^2 = 18n^2q^2$.

□

Theorem 4.3. *Let $\Gamma = \Gamma(U_{6n})$. Then, the independent polynomial is as follows:*

$$I(\Gamma; q) = 1 + \sum_{k=1}^{n} \left(\binom{2n}{k} + 3\binom{n}{k} \right) q^k + \sum_{k=n+1}^{2n} \binom{2n}{k} q^k.$$

Proof. By Theorem 2.3, $\alpha(\Gamma) = 2n$. Then, $I(\Gamma; q) = \sum_{k=0}^{2n} s_k q^k$. It is easy to see that $s_0 = 1$ since the only independent set of cardinality zero is the empty set. Moreover, we have three independent sets, Ω_1, Ω_2 and Ω_3, each of cardinality n and one independent set, Ω_4, of cardinality $2n$. Thus, there are $s_k = 3\binom{n}{k} + \binom{2n}{k}$ possibilities of independent sets of cardinality k for $1 \leq k \leq n$, and $s_k = \binom{2n}{k}$ possibilities of independent sets of cardinality k for $n < k \leq 2n$. Then, the result follows. □

Corollary 4.2. *Let $\Gamma = \Gamma(U_{6n})$. Then, the vertex-cover polynomial is as follows:*

$$\psi(\Gamma; q) = q^{5n} + \sum_{k=1}^{n} \left(\binom{2n}{k} + 3\binom{n}{k} \right) q^{5n-k} + \sum_{k=n+1}^{2n} \binom{2n}{k} q^{5n-k}.$$

Proof. It is a straightforward from Theorem 4.3 and by the result $\psi(\Gamma, q) = q^{n(\Gamma)} I(\Gamma, q^{-1})$ [2]. □

5 Conclusions

In this paper, some properties of the non-commuting graph of the group U_{6n} is presented. The general formula of the resolving polynomial of the non-commuting graph of the group U_{6n} are provided. In the last section of this paper, we also provided the detour index, eccentric connectivity, total eccentricity and independent polynomials of non-commuting graphs on U_{6n}.

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